MEC Fall 2019: Koch Snowflake

Questions adapted from Maru Sarazola’s Math Explorer’s Club module materials, Spring 2017

1. What happens with the notions of area and perimeter, if instead of the usual geometric figures (squares, triangles, circles, etc.), we consider one that allows for an infinite construction? The Koch snowflake can be constructed by starting with an equilateral triangle, then recursively altering each line segment as follows:

(i) Divide the line segment into three segments of equal length.
(ii) Draw an equilateral triangle that has the middle segment from step (i) as its base and points outward.
(iii) Remove the line segment that is the base of the triangle from step (ii).

Starting with an equilateral triangle, draw at least three iterations of steps (i)-(ii)-(iii). You might want to start with a large triangle! Of course, we can’t really draw the Koch snowflake, since its construction requires an infinite number of steps! When you get home, you might want to look up the Koch snowflake; Wikipedia (https://en.wikipedia.org/wiki/Kochsnowflake) has a great animation of what happens if you zoom in to the curve!

2. How many sides does the snowflake have before doing any iterations? After only one iteration of the process? After two? After three?

Triangle: 3; Iteration 1: 12; Iteration 2: 48; Iteration 3: 192 (each side turns into four line segments when iterating)

3. Write a formula for the number of sides of the snowflake after \(n\) iterations.

\(4^n(3)\)

4. Let’s assume that the sides of the equilateral triangle that you started with have length 1. What is the length of each side after one iteration? After two? After three?

Iteration 1: 1/3; Iteration 2: 1/9; Iteration 3: 1/27

5. Can you write a formula for the length of each side of the snowflake after \(n\) iterations?

\((1/3)^n\)

6. Combine your answers to Questions 3 and 5 to give a formula for the perimeter of the snowflake after \(n\) iterations.

Number of sides times length of each side = \(4^n(3)(1/3)^n\)

7. Using your intuition: What do you think the perimeter of the Koch snowflake will be, after it is fully constructed?

Infinity!

8. Can you think of a way to prove your answer for Question 7, in a way that doesn’t rely on intuition and that would convince even a skeptical mathematician?

The perimeter at any step is \((4/3)^n\).
9. Drawing with a magnifying glass: Suppose that now the sides of the equilateral triangle that you started with have length \( s \). Can you give a formula for the perimeter of the snowflake after \( n \) iterations?

Hint: It might help to go over your answers for Questions 4 and 5. What would the perimeter of the fully constructed snowflake be in this case? Does it matter if we make \( s \) smaller and smaller?

After \( n \) iterations: \( = 4^n (3)(1/3)^n \). Full snowflake: Infinity! It does not matter.

10. How many triangles are added in the first iteration of the process? In the second one? And the third?

   Iteration 1: 3; Iteration 2: 12; Iteration 3: 48.

11. How many triangles are added in the \( n \)-th iteration?

   One for each side of the previous iteration.

12. Let’s go back to assuming that the initial equilateral triangle had sides of length 1. What is the area of each triangle added in the first iteration of the process? In the second one? And the third?

   Iteration 1: \( (\frac{1}{3})^2 \sqrt{3}/4 \); Iteration 2: \( (\frac{1}{9})^2 \sqrt{3}/4 \); Iteration 3: \( (\frac{1}{27})^2 \sqrt{3}/4 \)

13. What is the area of each triangle added in the \( n \)-th iteration?

   \( (\frac{1}{3^n})^2 \sqrt{3}/4 \)

14. Combine Questions 11 and 13 to write a formula for the total new area added after the \( n \)-th iteration.

   \( (\frac{1}{3^n})^2 \sqrt{3}/4 \cdot 4^{n-1}(3) \)

15. Will the area of the fully constructed Koch snowflake be infinite or finite? Even if you can’t figure it out from the formula, there is an ingenious, “non-mathematical” way to answer this question!

   Finite! We can rewrite our formula as \( (\frac{1}{3})^{n-2} k \), where \( k \) is a constant.

16. Does your answer for Question 15 change if the sides of the equilateral triangle that you started with have lengths instead of 1? What if \( s \) is taken to be larger and larger?

   A factor of \( s^2 \) will be part of the constant, so as \( s \) increases the area will increase. However, no matter how large \( s \) is the area remains finite.

Summary: When we allow for geometric figures whose construction involves infinity in some way, our notions of area and perimeter are challenged. Strange things can happen, like with the Koch snowflake, which has a finite area enclosed in an infinite perimeter! This is not a contradiction with our knowledge so far, because these notions need to be re-defined to accommodate for this infinite scenario.

Bonus prompt for students: Can you create other “infinite” geometric figures that also have the strange property mentioned above?