These are a set of problems for Math Explorer’s club (Fall 2018). The topic is Mathematical Induction.
1. **Counting number of diagonals**

a. Let’s start with a square (a polygon with 4 sides).

   ![Square](image)

   How many diagonals are there in a square? (A diagonal is a line segment joining two different vertices of a polygon, when two vertices are not joined by an edge. Corner points are vertices.)

b. How would you count number of diagonals of a pentagon based on your answer in part a)?

   ![Pentagon](image)

   You can think about this in two ways:
   1. Directly count the number of diagonals.
   2. Use your count for number of diagonals of square to count the diagonals for pentagon.

Repeat the above exercise for a hexagon.

![Hexagon](image)

c. Proceeding in a similar way, can you make a guess about number of diagonals in a polygon with n sides?

Bonus question: Does the answer change if not all sides have the same length?
2. A tiling problem

Setup: We are given a $2 \times n$ grid.

Aim: To cover it using plain dominos or $2 \times 1$ tiles that look like this.

Let $f_n$ denote the number of ways to cover a $2 \times n$ grid using dominos.

a. What are $f_1$ and $f_2$? Below is a picture of a $2 \times 1$ and a $2 \times 2$ grid.

b. Count all the ways to tile a $2 \times 3$ grid using these dominoes.

c. Try counting $f_3$ in a different way. What is $f_3$ in terms of $f_1$ and $f_2$? (Start by tiling from leftmost tile).

d. Using a similar approach as in part (c), count $f_4$ and $f_5$.

Bonus: Make a guess for a formula of $f_n$ in terms of $f_{n-1}$ and $f_{n-2}$.
3. Demonstrating use of induction

Now that you have seen some problems which deal with "inductive arguments" and you have guessed some formulas for the number of diagonals and the number of ways of tiling. Let’s discuss about using technique of induction to prove a formula.

3.1. Proof using induction. Start with a statement $S(n)$ that depends on $n$ that you want to prove, (you can arrive at this statement through exploration, informed guessing or by some other method). These are the steps of induction.

1. **Base Step:** Check that your statement is true for $n=1$. Depending on problem, you can check this for $n=0$ or for some few small cases.
2. **Inductive Step:** Assume that $S(n-1)$ is true and prove that $S(n)$ is true.
3. **QED** You are all set!

3.2. Sum of natural numbers. Let $\{1, \ldots, n\}$ be the set of the first $n$ natural numbers. Let $S_n$ denote sum of these elements.

We can easily compute by hand that $S_1 = 1$, $S_2 = 3$, $S_3 = 6$ and so on.

Out of the following formula which one do you think is equal to $S_n$?

a. $2n + 1$

b. $n(n + 1)/2$

c. $n^2(n + 1)/2$

Once you have decided on what should be the formula, its time to prove your guess using induction.

1. Check that your formula is true for $n=1$.
2. Assume that $S_{n-1}$ is equal to your formula, is it true that $S_n$ is also equal to your formula? (Think about how $S_n$ and $S_{n-1}$ are related).

A hard question: What steps would you take to try to guess the sum of first $n$ squares, $1^2 + 2^2 + \ldots + n^2$?
3.3. **Fibonacci sequence.** The fibonacci sequence is one of the most interesting sequences of numbers I have come across. Here are some cool facts about Fibonacci sequence:

1. In the book, *The Da Vinci Code* (can you name the writer?) these numbers play a part in solving the mystery.
2. In the movie *Arrival* (2016), the character Ian Donnelly played by Jeremy Renner checks if the aliens have communicated with humans in any of the following approaches: "shapes, patterns, numbers, fibonacci".
3. (Spoiler Alert ahead!) The scientist character Walter Bishop in the television show *Fringe* recites the Fibonacci sequence to fall asleep. It is later revealed to be the key sequence identifying a series of safe deposit boxes he had maintained.
4. Fibonacci numbers appear unexpectedly often in mathematics, so much so that there is an entire journal dedicated to their study, the Fibonacci Quarterly. Applications of Fibonacci numbers include computer algorithms, biological settings, such as branching in trees, phyllotaxis (the arrangement of leaves on a stem), the fruit sprouts of a pineapple, the flowering of an artichoke, and so on.

The sequence is constructed as follows. The sequence starts with 1, 1 and the next term is obtained by adding the previous two terms.

So the sequence is 1, 1, 2, 3, 5, 8, ... and so on.

1. Write down first 10 terms of this sequence.
2. Is it true that every third term of this sequence is even? How do you know this?
3. Check that the sum of the first four Fibonacci numbers with odd indices is the eighth Fibonacci number (that is, check \( f_1 + f_3 + f_5 + f_7 = f_8 \)). Is it always true that the sum of the first k Fibonacci numbers with odd index is the \((2k)\)th Fibonacci number? If not, what’s a counter-example?
4. Suppose I say that the sum of the first k Fibonacci numbers with even index is the \((2k+1)\)th Fibonacci number. If this is true, try to prove it is so. If not, what should I say instead?
5. What recurrence relation do the terms satisfy? Does this relation looks familiar to you?

Bonus problem : From the relation, we can obtain a formula for \(n\)-th term of fibonacci sequence which is \( \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \sqrt{5} \) (its not easy to arrive at this formula!). Can you prove this using the technique of induction?
4. **Tower of Hanoi**

Setup: A model set of Tower of Hanoi with 9 disks.

Aim: To move the entire set of disks onto another rod, with the following rules.

1. Only one disk can be moved at a time.
2. Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack or on an empty rod.
3. No disk may be placed on top of a smaller disk.

a. Try to solve this puzzle when we have 3 disks. How many moves did you take?
b. Let’s say that your answer obtained in part a) is m. Can you solve the puzzle (for 3 disks) in less than m steps?
c. What do you think is the minimum number of steps required to complete this puzzle when we have 3 discs?
d. Now, that you have solved the puzzle for 3 discs. Think about how would you complete the puzzle for 4 discs?

Bonus question:

a. Let’s say that minimum number of moves required to solve this puzzle for n discs is \( T_n \) and for \( n + 1 \) discs is \( T_{n+1} \). Based on your observation in part d) discuss as to how you would write \( T_{n+1} \) in terms of \( T_n \).
5. Fun with divisibility

Let’s look at the sequence of numbers $11^n - 4^n$.

a. What is the first number in this sequence? Is it divisible by 7?

b. If your answer in part a) is yes, go ahead and check whether 7 divides the second and third number of the sequence.

c. If someone claims that 7 divides each number in this sequence, how would you verify this statement? (Proof by induction)

   1. You have verified this statement for the first term. This is called the Base step.

   2. Let’s assume that this statement is true for $(n - 1)$th term. Is it true for $n$th term? (inductive step)
6. A GAME WITH NUMBERS

Two people sit facing each other, call them Alexander and Kathleen; these are the players. A third person secretly writes two consecutive natural numbers on two slips of paper, and tapes each piece on the two players’ foreheads (one on each). The third person then leaves the room (or sits quietly); his role in the game is finished.

Alexander can see the number taped to Kathleen’s forehead, and likewise she can see the number taped to his forehead. So they both know the number that’s not their own. They also both know that the two numbers are consecutive.

One player, say Alexander, begins the game by asking Kathleen if she knows what her number is. If she does and she is correct, she says so and the game ends. If not, Alexander’s turn ends and Kathleen gets her chance to ask him if he knows his number. As before, if he does then he says so and the game ends. Otherwise, it becomes his turn again, and he repeats his original question to Kathleen. This back and forth questioning continues until someone finally says "Yes", if ever.

(We must assume that the two players are "perfect reasoners", so that if there was some way for either of them at any point to deduce their own number then they would do it. Without this assumption, whether or not the game ever ends might depend on whether or not one of the players is clever enough, and this kind of question doesn’t lend itself to a mathematical answer.)

Discussion problems:

a. Suppose that Kathleen’s number is 1. Will this game end?
b. Let’s say that Kathleen’s number is 3. Will this game end?
c. Now suppose that Kathleen’s number is 6. What do you think now? Will this game end?
d. Does this game ever end, no matter what the numbers are?
e. If this game ends (at all), then can you say something about number of steps in which the game should end?
7. Pirate Problem

There are 5 pirates (in strict order of seniority from highest rank to lowest rank A, B, C, D and E) who found 100 gold coins. They must decide how to distribute them. The pirate world’s rules of distribution say that the most senior pirate first proposes a plan of distribution. The pirates, including the proposer, then vote on whether to accept this distribution. If the majority accepts the plan or there is a tie, the coins are dispersed and the game ends. If the majority rejects the plan, the proposer is thrown overboard from the pirate ship and dies, and the next most senior pirate makes a new proposal to begin the system again. The process repeats until a plan is accepted or if there is one pirate left.

Pirates base their decisions on four factors. First of all, each pirate wants to survive. Second, given survival, each pirate wants to maximize the number of gold coins each receives. Third, each pirate would prefer not to throw another overboard, if all other results would otherwise be equal for that pirate (pirates have some heart after all), and finally, the pirates do not trust each other, and will neither make nor honor any promises between pirates apart from a proposed distribution plan that gives a whole number of gold coins to each pirate. (Remember 0 is a whole number!).

Discussion problem

a. What is the maximum amount of cash pirate A can pocket?
   For above problem, it would be helpful to think of this scenario when you have 2 pirates, then think about 3 pirates.

b. Let’s change the rules of this world. Suppose that tie votes result in the ousting of top pirate (or proposer). How would your answer change in (a) now?

c. How much cash pirate A can pocket if the members vote No rather than Yes if they get the same payoff either way? (Assume that tie results in ousting of the proposer).

d. How would your answer in part (c) change if a tie results in acceptance of plan?
8. Muddy Children Problem

Several children are playing together outside. After playing they come inside, and their mother says to them, at least one of you has mud on your head. Each child can see the mud on others but cannot see his or her own forehead. She then asks the following question over and over:

Can you tell for sure whether or not you have mud on your head? Assuming that all of the children are intelligent, honest, and answer simultaneously, what will happen? They cannot discuss among themselves.

a. Suppose that there is exactly one child with mud on their forehead. After the mother asks the question once, who will answer it first? Why others cannot answer it?

b. Suppose that there are exactly two children with mud on their forehead. What will happen in this case? How many times does the mother have to ask the question to hear an answer?

c. Discuss how would you build your argument inductively for 3 muddy children. Repeat this exercise for 4 and 5 muddy children.

d. Now that you have handled some specific cases, discuss about a general case of n muddy children.
9. A PROBLEM OF COINS

Suppose you have an infinite supply of 4 cent and 5 cent coins.

a. Is it true that 12 cents can be represented by 4’s and 5’s? What about 13, 14, 15?

b. If your friend makes a statement that, "I think all amounts after 12 can be obtained from 4’s and 5’s". How would you go about proving (or disproving) this statement. Think of using induction in your argument.
10. You have to be careful though!

By now, you must be convinced that induction is indeed a useful tool to mathematically prove your intuition or prove your guess about patterns. However, "with every great power comes great responsibility". Well, your responsibility at this stage is to be careful about using induction. We will demonstrate one scenario here.

10.1. **Statement**: All horses are the same colour. The case $n = 1$ is obvious: every group consisting of 1 horse is monochromatic.

Next comes the induction hypothesis: we are allowed to assume the result is true for $n$, and our job is to prove it true for $n + 1$.

So consider a group of $n + 1$ horses. How can we show that they are all the same color?

Well, our inductive hypothesis tells us that every group of $n$ horses is monochromatic. So all we have to do is remove one horse, call him Paul, from our group of $n + 1$ horses and consider the remaining group; call it $G_1$. There are $n$ horses left in $G_1$, so by the inductive hypothesis they are all the same color.

Now we have the problem that Paul is perhaps a different color than the rest. But this problem too can be overcome: simply remove a different horse from the original group. This leaves behind a new group of $n$ horses which includes Paul; call it $G_2$. Again we apply our inductive hypothesis, this time to deduce that $G_2$ is monochromatic. But if Paul is the same color as all the horses in $G_2$, then Paul must be the same color as a horse in $G_1$, since every horse in $G_2$ except Paul is also in $G_1$. This then shows that Paul is the same color as every horse in $G_1$ (since $G_1$ is monochromatic), and so the original group of $n + 1$ horses is monochromatic.

This completes the induction and thereby proves that all horses are the same color.

At first glance, this seems a perfectly good proof. Obviously it is wrong (Is it?). Discuss the argument once again to see what went wrong in the proof. Is induction not a reliable method? Can you use same argument to prove that every tree is of same colour?

**References**


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