In this worksheet, we will be discussing Cantor sets, which are a family of interesting mathematical objects. These are a class of infinite sets that have several unusual properties, which we will be exploring below.

1. We will begin by constructing the standard Cantor set. Here are the steps to creating it: be sure to draw along and draw a new picture at each step!
   
   (i) Draw the part of the number line from 0 to 1.
   (ii) Now redraw the number line from above without the numbers between 1/3 and 2/3.
   (iii) Now redraw the number line from the previous step, but without the numbers between 1/9 and 2/9 and without 7/9 to 8/9. How many pieces does your picture have?
   (iv) Following the pattern above, draw the next step. How many pieces does this picture have?

   At some point throughout this worksheet, one can talk about interval notation and use some of these constructions as a game to practice using that terminology.

2. Imagine that you kept drawing the pictures from the previous question for many more steps. What do you think happens? How many pieces are there at each step?
   The number of pieces at each step is a power of 2. The sizes of the remaining intervals get smaller.

3. If we repeated this construction infinitely many times, is anything left? If so, what does it look like? Can you name any numbers that remain? We call what is left after doing this infinitely many times the Cantor set.
   The only obvious things that are left are the endpoints of each interval; it is probably easiest to get the students to realize 0 and 1 are left. One can then see if the students can realize 1/3 also remains, etc. Do the students think there are any intervals left?
   *Item to discuss:* What size of infinity do we think the Cantor set might be? One can tie this into the Hilbert Hotel worksheet. The Cantor set is uncountable, and we can give hints about ternary expansions if kids seem interested or if there is extra time.

4. We are now going to think about the length of the Cantor set. Go back and look at your drawings from parts (i)-(iv) of Question 1. For each drawing, find the length of each piece. Add together the lengths of all pieces in the same drawing.

5. Can you find a pattern in the lengths you found for each drawing? What do you think happens to the length if you repeat step (iv) many times? What about infinitely many times? What do you think the length of the Cantor set should be?
   The lengths follow the pattern \((2/3)^{n-1}\), based on how the steps were numbered above. The length of the Cantor set is 0.

6. We’re now going to make a different kind of Cantor set, which is sometimes called the Fat Cantor set. Below are the steps to constructing it:
   
   (i) Draw the part of the number line from 0 to 1.
   (ii) Now redraw the number line from above without the numbers between 3/8 and 5/8. What was the length of the piece we removed?
(iii) Now redraw the number line from the previous step, without the numbers between 5/32 and 7/32 and the numbers between 25/32 and 27/32. What was the length of each piece we removed here?

(iv) Following the pattern above, draw the next step. Hint: What length should the intervals we removed be? How many intervals should we remove?

Here we are removing the middle interval of length $2^{-2k}$ at step $k$, if we count the first step above as step 0. The students should be able to figure out we remove intervals of length 1/4, 1/16, 1/64, \ldots. The students should also be able to figure out how many pieces are in each step from above.

7. Imagine we repeat the steps above infinitely many times. Is anything left? If so, try to describe what is left. We call what remains the Fat Cantor set.

The students should realize that once again endpoints are left. The answer to this is pretty much the same as for the standard Cantor set.

8. What do you think the length of the Fat Cantor set might be?

Depending on time, children can be given significant hints here. Key point: It is 1/2! One can see this by computing the lengths removed at each step and looking at a geometric series (note that one removes multiple intervals at each step) that sums up to 1/2 and/or by computing the remaining lengths, which also tend toward 1/2.

9. Create your own Cantor set!

There are many variations here! It should be noted Cantor sets can be created that have measure (length, as we were saying here) $\lambda$ for any $\lambda \in [0, 1)$. Students can try to compute the length of the sets they or their peers design; they can also do this whole activity together in groups at chalkboards if available.