Questions adapted from Maru Sarazola’s Math Explorer’s Club module materials, Spring 2017

1. You are the manager of Hilbert’s Hotel, a one of a kind establishment renowned for having infinitely many rooms, numbered 1, 2, 3, 4, . . . . When taking the job, you pledged to never turn down a guest, no matter what! Luckily for business, the hotel is completely booked, with guests occupying every single room.

Unexpectedly, one more guest turns up! Knowing your hotel’s reputation, he didn’t bother to make a reservation, and now he’s requesting a room. Your guests are in a really good mood – they’re on vacation after all! – and they are all willing to change rooms once, as long as you tell each of them explicitly the number of the room they should go to. In other words, you should be able to tell the guest staying in room number \( n \) to which room number \( m \) they have to go to.

How can you accommodate the newcomer?

One possibility (there are infinitely many!): Put the newcomer in room 1 and for all \( n \), move the person in room \( n \) to room \( m = n + 1 \)

2. What if 50 people arrived at once, requesting separate rooms? Can you accommodate all of them?

Put the new people in rooms 1 - 50 and for all \( n \), move the person in room \( n \) to room \( m = n + 50 \)

3. What if \( k \) people arrived at once, for some positive integer \( k \)?

Put the newcomers in rooms 1 - \( k \) and for all \( n \), move the person in room \( n \) to room \( m = n + k \)

4. Given your huge success, you decide to expand the services of the hotel to include transportation from the nearest airport. To carry on with the tradition, you fund Hilbertline, the first line of buses with infinitely many seats! When you buy the ticket, you are assigned a seat number from the infinite list of seats 1, 2, 3, 4, . . . .

Your hotel is still full to capacity when the first of your buses arrives. The bus is also full (that means it’s carrying infinitely many passengers!). How will you house your new guests, if they all want to be in separate rooms?

Label each person on the bus by their seat number \( s \). Put the person in seat \( s \) into room 2\( s \) in the hotel. Move the person in room \( n \) to room 2\( n - 1 \).

5. Your hotel is full once more, and just when you think you have a moment to rest, three of your buses arrive at the same time, all of them full with an infinite number of tourists who each want a room to themselves. What can you do now?

Label each person on the bus by their seat number \( s \). Put the person on bus 1 in seat \( s \) into room 4\( s \) in the hotel. Put the person on bus 2 in seat \( s \) into room 4\( s - 1 \). Put the person on bus 3 in seat \( s \) into room 4\( s - 2 \). Finally, have the person in room \( n \) move to room 4\( n - 3 \).

6. What would you do if instead of three, you had \( k \) full buses arriving at the same time?

Label each person on the bus by their seat number \( s \). Put the person on bus number \( b \) in seat \( s \) into room \((k + 1)s - (b - 1)\) in the hotel. Then have the person in room \( n \) move to room \((k + 1)n - k\).

7. A manager’s headache. What if you had infinitely many full buses arriving at the same time: Bus 1, Bus 2, Bus 3, Bus 4 . . . ?

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Assign odd prime numbers to each of the buses, so bus one is the first prime (3), bus two is the second prime (5), and so on (let us associate bus \( k \) with the \( k \)-th odd prime, which we will call \( p_k \)). Then send the person in seat \( s \) on bus \( k \) to room \( p_k^s \). Send the person already in room \( n \) to room \( 2^n \).

At this point the instructor(s) can discuss the fact that an infinite set can be mapped to a smaller subset in such a way that no two elements of the original set map to the same element in the image (injections) – this is what the students have been creating.

As a follow up to this, they can then ask the kids to come up with some examples of injections, taking the set of positive integers \((1,2,3,4,\ldots)\) as their starting infinite set.

8. Time to call it quits. There’s going to be a huge rock concert nearby, and you make sure the hotel is completely empty by the time the show is done, so nobody needs to move at night. However, the groupies are math fanatics, and they agreed each of them would wear a T-shirt with a decimal number between 0 and 1 on it, so that no two of them have the same number and that every single number between 0 and 1 is displayed on somebody’s T-shirt.

You try to give each person one room... but you can’t! Can you prove why?

There are more real numbers than natural numbers. One can get into countable versus uncountable and different sizes of infinity.